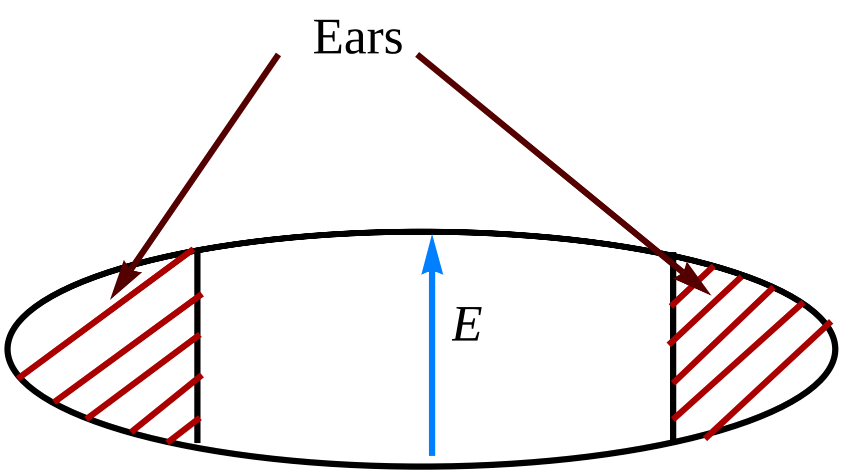
Simple Calculation To See Whether "Ears" will work

After talking to Bob on Tuesday 01 Feb 2011, I wanted to understand how the mode just above cutoff can be affected by the change of geometry. This question came up because I/we weren't sure why putting in obstructions in the long edges (Bob calls them ears) will actually affect the cut off frequency for a vertically polarized E-field, i.e.



I decided, after that conversation, to see if there's an easier way to understand whether "ears" will work (or not). Talking to my office mate, Ding Sun, he suggested that one way to understanding the problem is to use a rectangular waveguide. I thought it was a good idea because we know how to solve rectangular waveguides while for the elliptical waveguide especially with ears, it is probably not solvable analytically.

So the idea is to approximate the MI elliptic waveguide with a rectangular waveguide. Then by changing the size of the walls, I can calculate whether it is ok to use a vertically polarized E-field or we have to use a horizontally polarized E-field solution. So here goes ...

MI elliptical pipe approximated with rectangular pipe

Looking at Paul's paper ("Computation of electron cloud diagnostics and mitigation in the main injector", SciDAC2009, doi: 10.1088/1742-6596/180/1/012007), the MI beampipe ellipse has semi-major axis = 5.8801 cm (2.315") and semi-minor axis = 2.3876 cm (0.94").

In[1]:=
```
MIra = 5.8801 × 10^-2; (*m semi-major axis*)
```

In[2]:=
```
MIrb = 2.3876 × 10^-2; (*m semi-minor axis*)
```

Size of the rectangular pipe

In the rectangle beam pipe approximation, I will make the constraint that the rectangle has the same cross sectional area as the ellipse. (Of course, there may be better ways to approximate the rectangle. This is what popped into my head right now). Area of ellipse is

In[3]:=
```
MIarea = π MIra MIrb (*m^2*)
```

Out[3]= 0.00441058

If I choose the long size of the rectange to be the major axis of the ellipse, then I have

In[4]:=
```
La = 2 MIra (*m*)
```

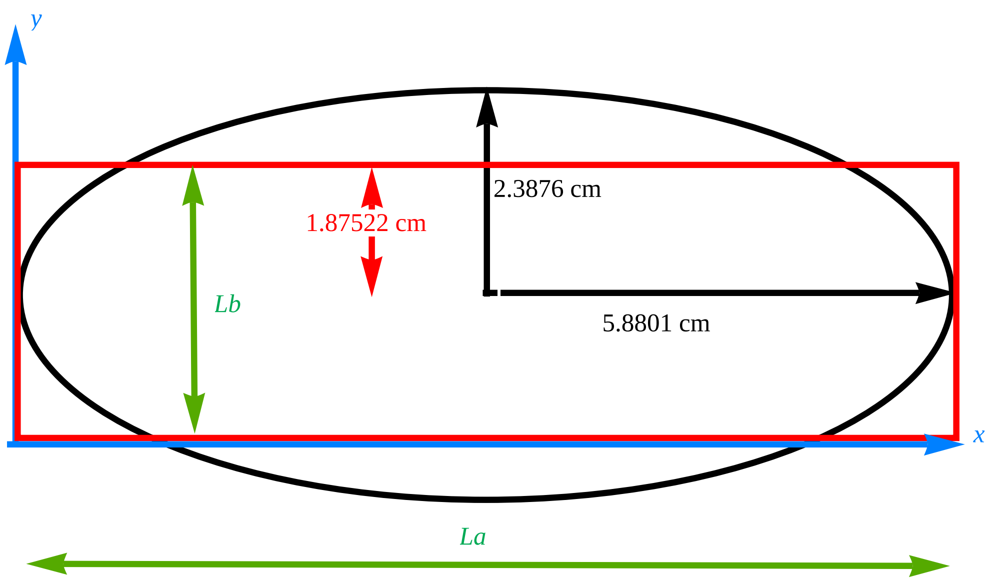
Out[4]= 0.117602

Therefore, the short side is

In[5]:=
```
Lb = MIarea / La (*m*)
```

Out[5]= 0.0375043

Here's how the approximation looks like with the elliptic beampipe superimposed:



The coordinate system is as shown above. The bottom left corner of the waveguide is at (0,0).

Modes of the Rectangular Waveguide

I am using the formulas from “Fields and Waves in Communication Electronics”, S. Ramo et al. The lowest mode in a rectangular waveguide is the TE10 mode which has a cutoff frequency given by

$$fc = \frac{1}{2a\sqrt{\mu\epsilon}} = \frac{c}{2a}$$

where a is the length of the x axis, i.e. **independent** of the length of the waveguide in the y axis. The TE10 mode has E-field polarization parallel to the y axis.

The cutoff frequency for any arbitrary mode is (whether it is TM or TE). Section 8.7, eq(7)

In[6]:=
```
fc[m_, n_, a_, b_] := c/(2 π) Sqrt[(m π/a)^2 + (n π/b)^2]
```

$$fc[m\_, n\_, a\_, b\_] := \frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

where (m,n) is the mode number. a is the length of the waveguide along the xaxis, b is the length of the waveguide in the yaxis

In[7]:=
```
c = 3 × 10^8; (*m/s*)
```

Therefore, for the TE10 mode, the cut off frequency for my rectangular waveguide is

In[8]:=
```
f10 = fc[1, 0, La, Lb]
```

Out[8]= 1.27549×10^9

or approximately 1.28 GHz

The next highest mode is the TE01 mode which has E-field polarized parallel to the x-axis and has a cutoff frequency

In[9]:=
```
f01 = fc[0, 1, La, Lb]
```

Out[9]= 3.99954×10^9

This is REALLY far away from f10 in this approximation. Therefore, f10 is the way to go. Note TE01 and TE20 are degenerate if La = 2 Lb modes because they have the same cutoff frequency.

■ Plotting out the Magnitude of the E - fields

The E-field solution for the rectangular waveguide is

$$Ex \sim Cos\left[\frac{m\pi x}{a}\right] Sin\left[\frac{n\pi y}{b}\right]$$

$$Ey \sim Sin\left[\frac{m\pi x}{a}\right] Cos\left[\frac{n\pi y}{b}\right]$$

Therefore, for TE10, Ex = 0, and Ey~$Sin\left[\frac{\pi x}{a}\right]$. And similarly, TE01, Ex~$Sin\left[\frac{\pi y}{b}\right]$, i.e. orthogonal to TE10.

In order to plot this out, I am going to just assume that Ex = Cos[$\left[\frac{m\pi x}{a}\right]$ Sin$\left[\frac{n\pi y}{b}\right]$ and Ey = Sin$\left[\frac{m\pi x}{a}\right]$ Cos$\left[\frac{n\pi y}{b}\right]$

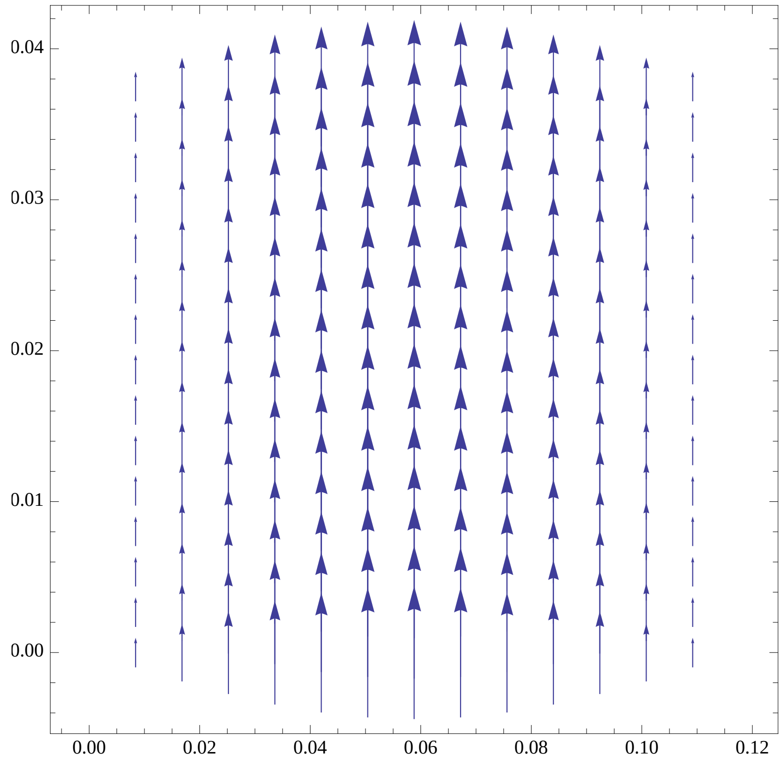
In[10]:=
```
Ex[m_, n_, x_, y_] := Cos[m π x/La] Sin[n π y/Lb];
```

In[11]:=
```
Ey[m_, n_, x_, y_] := Sin[m π x/La] Cos[n π y/Lb];
```

■ For TE10

In[12]:=
```
VectorPlot[{Ex[1, 0, x, y], Ey[1, 0, x, y]}, {x, 0, La}, {y, 0, Lb}]
```

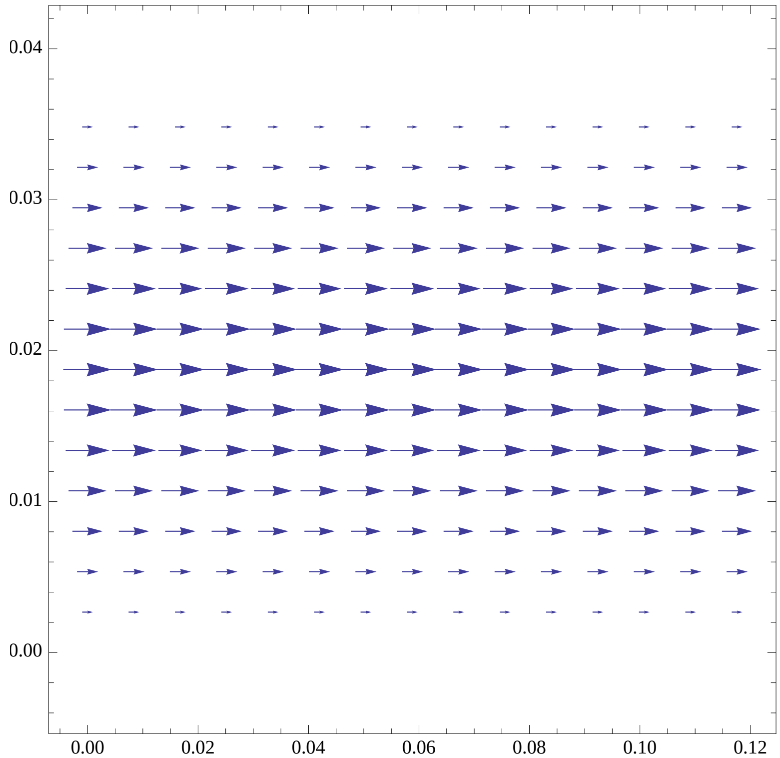
Out[12]=



- **For TE01**

In[13]:=
```
VectorPlot[{Ex[0, 1, x, y], Ey[0, 1, x, y]}, {x, 0, La}, {y, 0, Lb}]
```

Out[13]=

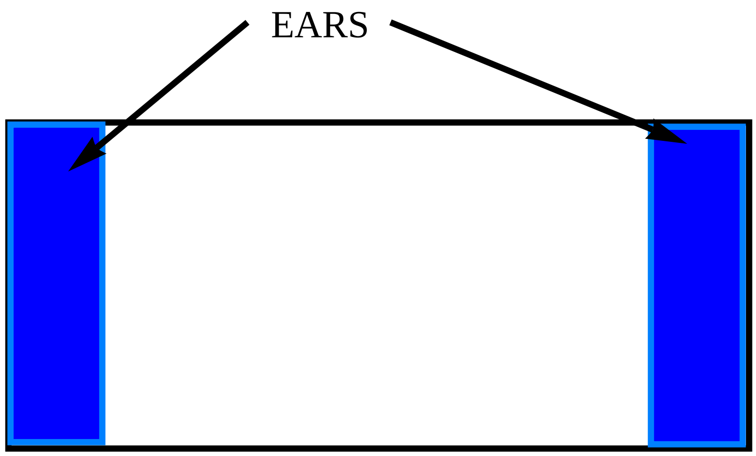


Putting in Ears

Since the waveguide is rectangular, putting in ears is equivalent to increasing the cutoff frequency of the TE10 mode. Suppose I decrease La by 1cm, i.e. the ears are 0.5cm, then the cutoff frequency is

In[14]:=
```
fEARSc = fc[1, 0, La - 10^-2, Lb]
```

Out[14]= 1.39403×10^{9}



EARS are 0.5 cm in length and cut down the length of the waveguide by 1cm to 10.76 cm

Reflection Coefficient

I can calculate the reflection coefficient R by calculating the impedance of TE10 in the “cavity” and in the “eared” region. I want Z to be REAL in both regions because I want R to be real, therefore, the transmitter frequency must start from at the cutoff frequency of the eared region fEARSc. (See my previous BOE “Measuring Electron Cloud Density with Trapped Modes” in beamdocs).

From Ramo section 8.8 eq(7), the impedance of the TE10 mode is

In[15]:=
$$\text{ZTE}[\text{fc\_}, \text{f\_}] := \frac{\eta}{\sqrt{1 - \left(\frac{\text{fc}}{\text{f}}\right)^2}}$$

The reflection coefficient is given by

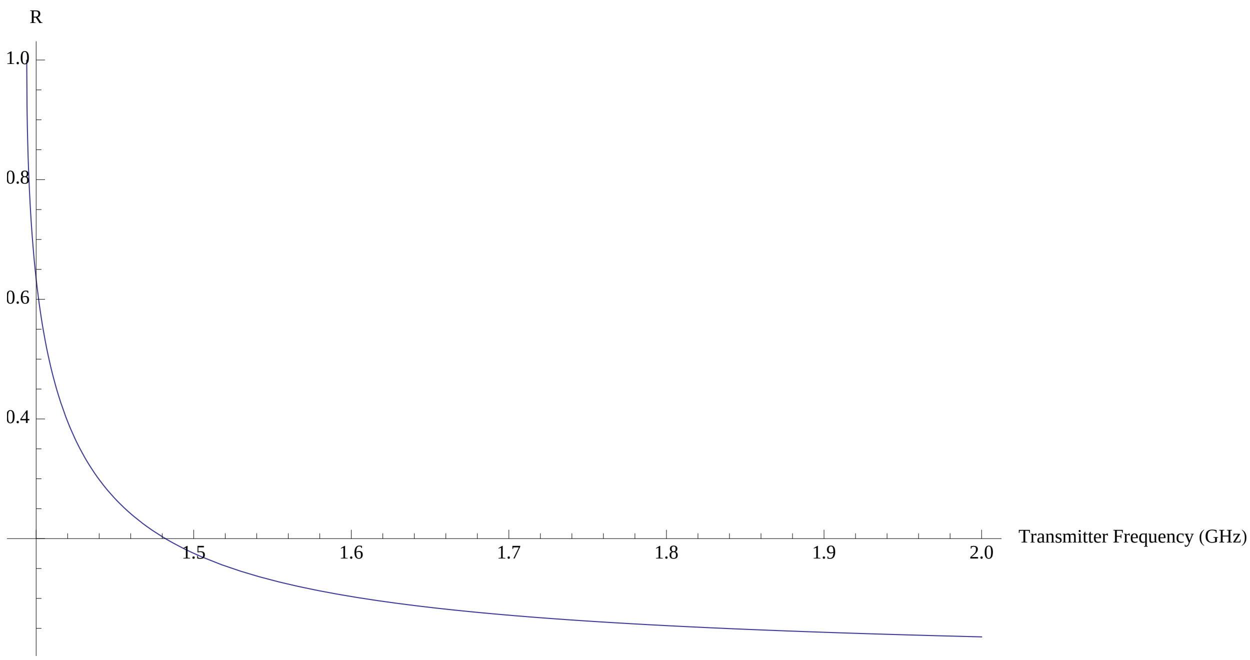
$$R = \frac{\text{Zears} - \text{Zcav}}{\text{Zears} + \text{Zcav}}$$

In[16]:=
$$\text{Rcoeff}[\text{f\_}] := \frac{\text{ZTE}[\text{fEARSc}, \text{f}] - \text{ZTE}[\text{f10}, \text{f}]}{\text{ZTE}[\text{fEARSc}, \text{f}] + \text{ZTE}[\text{f10}, \text{f}]}$$

▪ Plot

In[17]:=
```
Plot[Rcoeff[f 10^9], {f, fEARSc 10^-9, 2},
  AxesLabel → {"Transmitter Frequency (GHz)", "R"}, PlotRange → All]
```

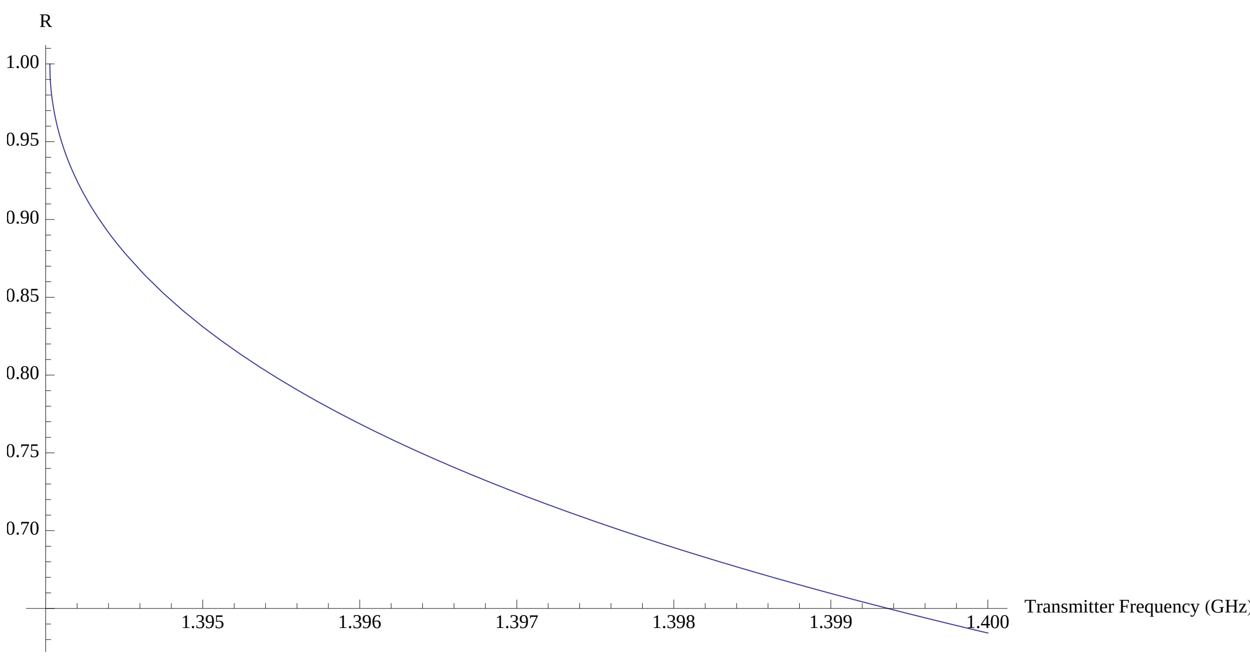
Out[17]=



▪ Zoom in

In[18]:=
```
Plot[Rcoeff[f 10^9], {f, fEARSc 10^-9, 1.4},
  AxesLabel → {"Transmitter Frequency (GHz)", "R"}, PlotRange → All]
```

Out[18]=



For a reflection coefficient of 0.95, the transmission frequency should be

```
In[19]:= fsol = FindRoot[Rcoeff[f 10^9] == 0.95, {f, 1.4}]
Out[19]= {f → 1.3941}
```

Therefore, the transmission frequency should be about 1.394 GHz which is about

```
In[20]:= Δf = f 10^9 - fEARSc /. First[fsol]
Out[20]= 74667.6
```

which is only 75 kHz above cutoff of the eared beam pipe.