An Estimate of Orbit Correction for SSR0 section of the Project X

Relationship between beta-functions, period length and phase advance

$$\sin\left(\frac{\mu}{2}\right) = \sqrt{\frac{L}{4\cdot F}}$$

$$\beta_{max} = \frac{L}{\sin(\mu)}$$

For μ = π/2 one has:

$$\beta_{max} = L \qquad F = \frac{L}{2}$$

Derivation

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{2\cdot F} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\frac{1}{2\cdot F} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{2\cdot F} & L \\ \frac{L - 4\cdot F}{4\cdot F^2} & 1 - \frac{L}{2\cdot F} \end{pmatrix} = \begin{bmatrix} \cos(\mu) & \beta_{max}\cdot\sin(\mu) \\ -\left(\frac{\sin(\mu)}{\beta_{max}}\right) & \cos(\mu) \end{bmatrix}$$

$$\cos(\mu) = 1 - \frac{L}{2\cdot F} = 1 - 2\cdot\sin\left(\frac{\mu}{2}\right)^2 \quad \Rightarrow \quad \sin\left(\frac{\mu}{2}\right) = \sqrt{\frac{L}{4\cdot F}}$$

$$\beta_{max} = \sqrt{\frac{4\cdot F^2\cdot L}{4\cdot F - L}} = \sqrt{\frac{4\cdot F^2\cdot\frac{L}{4\cdot F}}{1 - \frac{L}{4\cdot F}}} = F\cdot\sqrt{\frac{4\cdot\sin\left(\frac{\mu}{2}\right)^2}{1 - \sin\left(\frac{\mu}{2}\right)^2}} = 2\cdot F\cdot\tan\left(\frac{\mu}{2}\right)$$

$$\Rightarrow \quad \beta_{max} = \frac{L}{2\cdot\sin\left(\frac{\mu}{2}\right)^2}\cdot\tan\left(\frac{\mu}{2}\right) = \frac{L}{\sin(\mu)}$$

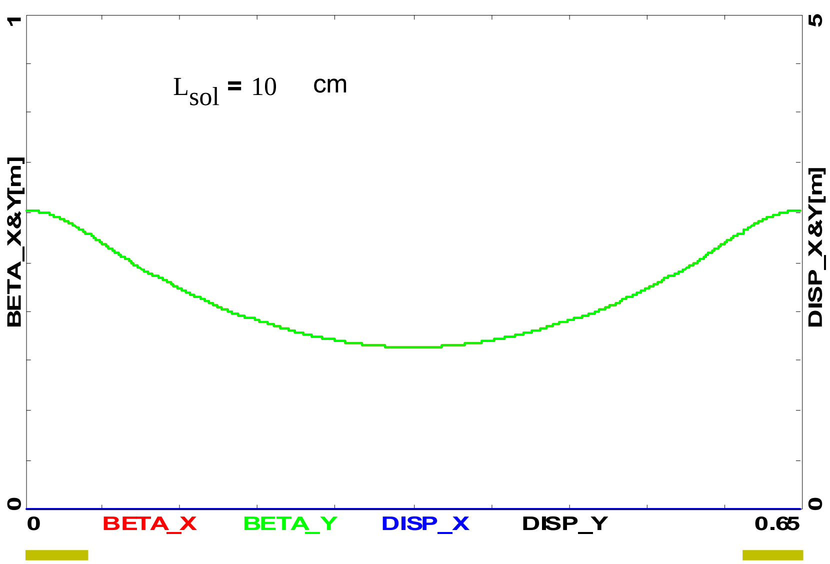
Focusing strength and maximum beta for SSR0:

$L := 65$ cm $\qquad \mu := \frac{\pi}{2} \qquad \beta_{max} := \frac{L}{\sin(\mu)}$

$F := \frac{L}{4\cdot\sin\left(\frac{\mu}{2}\right)^2}$

$F = 32.5$ cm $\qquad \beta_{max} = 65$ cm

Thu Apr 15 22:48:26 2010 OptiM - Untitled



Non-linearity of solenoidal focusing

Magnetic fields

$$B_z(z,\rho) = B_z(z) - \frac{\rho^2}{4}\cdot\frac{d^2}{dz^2}B_z(z) + \frac{\rho^4}{64}\cdot\frac{d^4}{dz^4}B_z(z) + O\left(\rho^6\right)$$

$$B_r(z,\rho) = -\frac{\rho}{2}\cdot\frac{d}{dz}B_z(z) + \frac{\rho^3}{16}\cdot\frac{d^3}{dz^3}B_z(z) - \frac{\rho^5}{384}\cdot\frac{d^5}{dz^5}B_z(z) + O\left(\rho^7\right)$$

Focusing in the first order of perturbation theory

$$\Phi(\rho) = \frac{1}{F(\rho)} = \frac{e}{2\cdot p\cdot c}\cdot\int_{-\infty}^{\infty}\left[B_z(z)^2 + \frac{\rho^2}{2}\cdot\left(\frac{d}{dz}B_z(z)\right)^2 + \frac{5\cdot\rho^2}{64}\cdot\left(\frac{d^2}{dz^2}B_z(z)\right)^2 + O\left(\rho^6\right)\right]dz$$

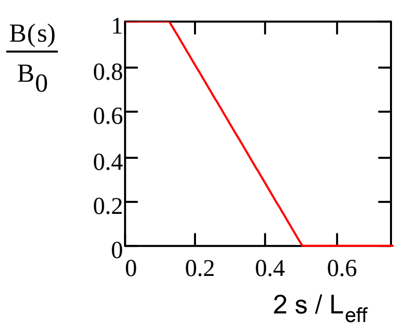
Leaving only the first nonlinear term one obtains:

$$\frac{\Delta\Phi(\rho)}{\Phi} = \frac{16}{3}\cdot\left(\frac{\rho}{L_{eff}}\right)^2$$

where the solenoid efefctive length is

$$\frac{1}{L_{eff}^2} = \frac{3}{32}\cdot\frac{\int_{-\infty}^{\infty}\left(\frac{d}{dz}B_z(z)\right)^2 dz}{\int_{-\infty}^{\infty}B_z(z)^2\,dz}$$

For the optimal case of linearly changing magnetic field L_{eff} coincides with the total field length



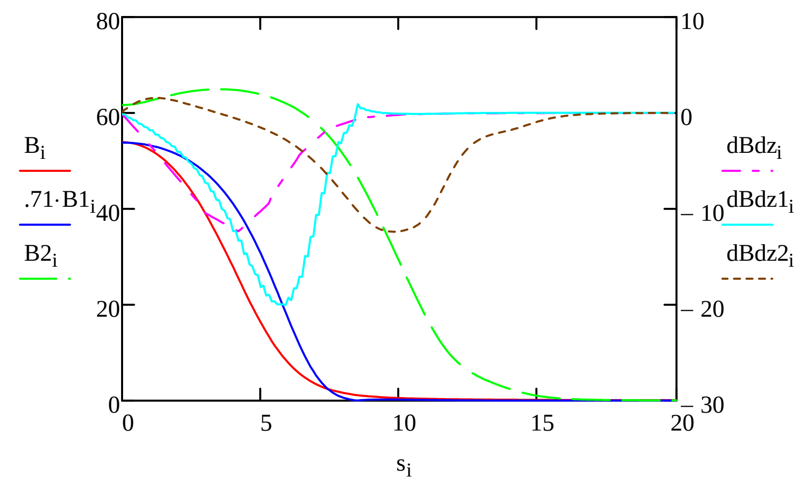
Reading the data

M := ACD\Bz_z_Lenses.xls   M1 := ACD\Bz_z_Lenses.xls   M2 := ACD\Bz_z_Lenses.xls

$i := 0..200$ $B_i := 10\cdot M_{i+4,2}$ $B1_i := 10\cdot M1_{i+4,2}$ $B2_i := 10\cdot M2_{i+5,2}$

$j := 0..199$ $s_i := 100\cdot M_{i+4,1}$

$$dBdz1_j := \frac{B1_{j+1} - B1_j}{s_{j+1} - s_j} \qquad dBdz2_j := \frac{B2_{j+1} - B2_j}{s_{j+1} - s_j} \qquad dBdz_j := \frac{B_{j+1} - B_j}{s_{j+1} - s_j}$$



$$f_B\left(s,L_{tot}\right) := \left(s < \frac{1}{4}\cdot\frac{L_{tot}}{2}\right) + \left(s \geq \frac{1}{4}\cdot\frac{L_{tot}}{2}\right)\cdot\left(s < \frac{L_{tot}}{2}\right)\cdot\left(1 - \frac{8\cdot s - L_{tot}}{3\cdot L_{tot}}\right)$$

$$L_{eff0} := \sqrt{\frac{32}{3}\cdot\sum_j\left[\left(B_j\right)^2\cdot\left(s_{j+1} - s_j\right)\right]\cdot\left[\sum_j\left[\left(dBdz_j\right)^2\cdot\left(s_{j+1} - s_j\right)\right]\right]^{-1}} \qquad \frac{L_{eff0}}{2} = 7.337 \text{ cm}$$

$$L_{eff1} := \sqrt{\frac{32}{3}\cdot\sum_j\left[\left(B1_j\right)^2\cdot\left(s_{j+1} - s_j\right)\right]\cdot\left[\sum_j\left[\left(dBdz1_j\right)^2\cdot\left(s_{j+1} - s_j\right)\right]\right]^{-1}} \qquad \frac{L_{eff1}}{2} = 7.878 \text{ cm}$$

$$L_{eff2} := \sqrt{\frac{32}{3}\cdot\sum_j\left[\left(B2_j\right)^2\cdot\left(s_{j+1} - s_j\right)\right]\cdot\left[\sum_j\left[\left(dBdz2_j\right)^2\cdot\left(s_{j+1} - s_j\right)\right]\right]^{-1}} \qquad \frac{L_{eff2}}{2} = 12.953 \text{ cm}$$

Numerical integration for solenoidal focusing

Measured magnetic field on axis is fitted by analytical expression

$a := 2.45 \quad s0 := 5.36$

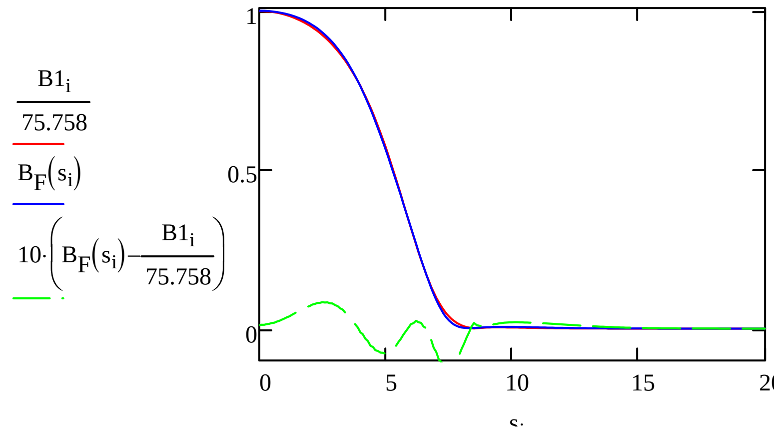
$$B_F(s) := \frac{1}{75.758}\cdot\left[\frac{-77.8}{2}\cdot\left(\left(\tanh\left(\frac{s-s0}{a}\right)-\tanh\left(\frac{s+s0}{a}\right)\right)\right)-9.8\cdot\left(\frac{1}{\cosh(s+7.3)}+\frac{1}{\cosh(s-7.3)}\right)\right]$$

Then the field outside of the axis is:

$$B_x(x,y,z) := -\frac{x}{2}\cdot\frac{d}{dz}B_F(z)+\frac{x\cdot\left(x^2+y^2\right)}{16}\cdot\frac{d^3}{dz^3}B_F(z)-\frac{x\cdot\left(x^2+y^2\right)^2}{384}\cdot\frac{d^5}{dz^5}B_F(z)$$

$$B_y(x,y,z) := -\frac{y}{2}\cdot\frac{d}{dz}B_F(z)+\frac{y\cdot\left(x^2+y^2\right)}{16}\cdot\frac{d^3}{dz^3}B_F(z)-\frac{y\cdot\left(x^2+y^2\right)^2}{384}\cdot\frac{d^5}{dz^5}B_F(z)$$

$$B_z(x,y,z) := B_F(z)-\frac{x^2+y^2}{4}\cdot\frac{d^2}{dz^2}B_F(z)+\frac{\left(x^2+y^2\right)^2}{64}\cdot\frac{d^4}{dz^4}B_F(z)$$

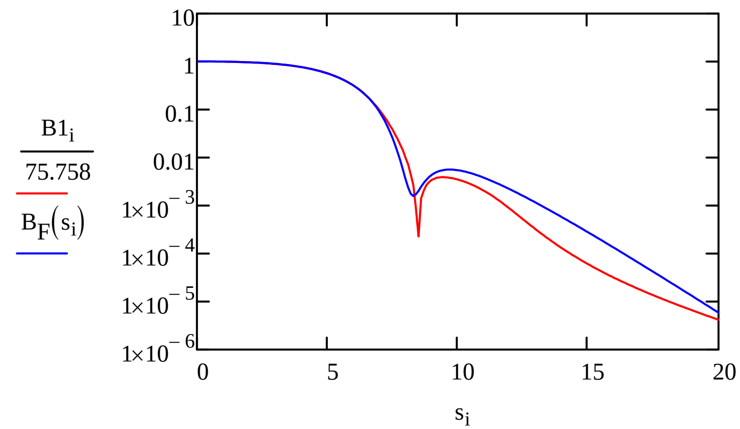


Equations to be solved are:

$$\frac{d^2}{d\tau^2}r = \frac{e\cdot B_0}{m\cdot c^2\cdot\beta\cdot\gamma}\cdot\frac{dr}{d\tau}\times\frac{B}{B_0} \qquad \kappa_B = \frac{e\cdot B_0}{m\cdot c^2\cdot\beta\cdot\gamma} \qquad \tau = \beta\cdot c\cdot t$$

or in the vector form

$$\frac{d}{d\tau}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ p_z \end{pmatrix} = \begin{bmatrix} p_x \\ \kappa_B\cdot\left(y\cdot B_z\left(Y_0,Y_2,Y_4\right)-z\cdot B_y\left(Y_0,Y_2,Y_4\right)\right) \\ p_y \\ -\kappa_B\cdot\left(x\cdot B_z\left(Y_0,Y_2,Y_4\right)-z\cdot B_x\left(Y_0,Y_2,Y_4\right)\right) \\ p_z \\ \kappa_B\cdot\left(x\cdot B_y\left(Y_0,Y_2,Y_4\right)-y\cdot B_x\left(Y_0,Y_2,Y_4\right)\right) \end{bmatrix}$$



Correspondting Mathcad implementation is:

```
GetFocus(x,z,px,py) := | for  i ∈ 0 .. rows(x) − 1
                       |   break  if  x_i < 0
                       |   F_0 ← z_(i−1) − (z_i − z_(i−1))/(x_i − x_(i−1)) · x_(i−1)
                       |   F_1 ← −atan2(py_i, px_i)
                       | F
```

$\theta_0 := 0.024 \qquad \kappa_B := 0.124$

$N_p := 200 \qquad j := 0..N_p$

$S1 := \mathrm{rkfixed}\left(Y_0, 0, 2.5\cdot L, N_p, D\right)$

$x := S1^{\langle 1\rangle} \qquad y := S1^{\langle 3\rangle} \qquad z := S1^{\langle 5\rangle}$

$px := S1^{\langle 2\rangle} \qquad py := S1^{\langle 4\rangle} \qquad pz := S1^{\langle 6\rangle}$

$$Y_0 := \begin{pmatrix} 0 \\ \theta_0 \\ 0 \\ 0 \\ -L \\ \sqrt{1-\theta_0^2} \end{pmatrix}$$

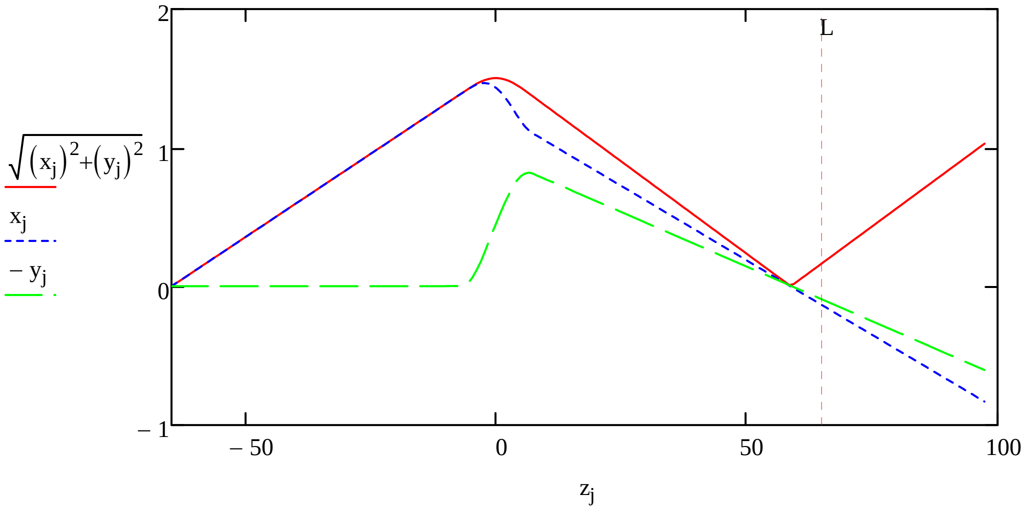
$$D(s,Y) := \begin{bmatrix} Y_1 \\ \kappa_B\cdot\left(Y_3\cdot B_z\left(Y_0,Y_2,Y_4\right)-Y_5\cdot B_y\left(Y_0,Y_2,Y_4\right)\right) \\ Y_3 \\ -\kappa_B\cdot\left(Y_1\cdot B_z\left(Y_0,Y_2,Y_4\right)-Y_5\cdot B_x\left(Y_0,Y_2,Y_4\right)\right) \\ Y_5 \\ \kappa_B\cdot\left(Y_1\cdot B_y\left(Y_0,Y_2,Y_4\right)-Y_3\cdot B_x\left(Y_0,Y_2,Y_4\right)\right) \end{bmatrix}$$

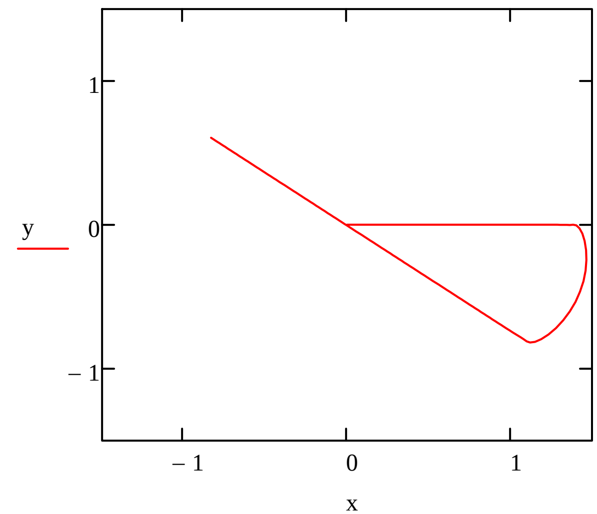
Comparison of Numerical Integration with Perturbation Theory

$F := \mathrm{GetFocus}(x, z, px, py)$

$F_0 = 58.582$ cm

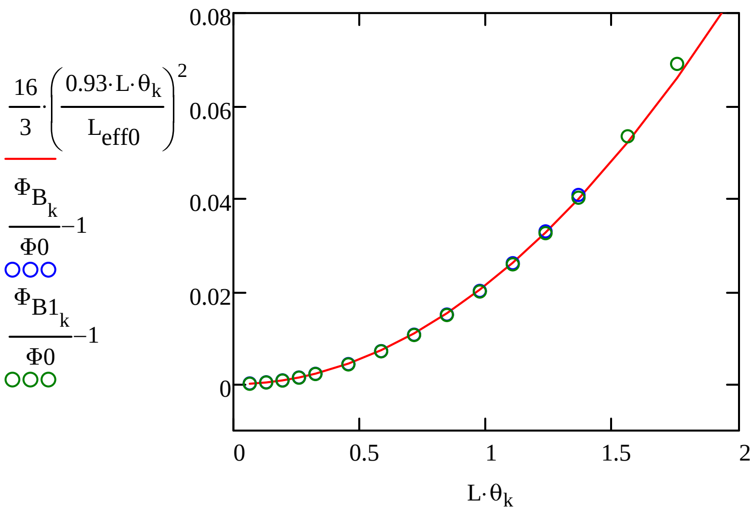
$\frac{\pi}{2} - F_1 = 36.23259\cdot deg$

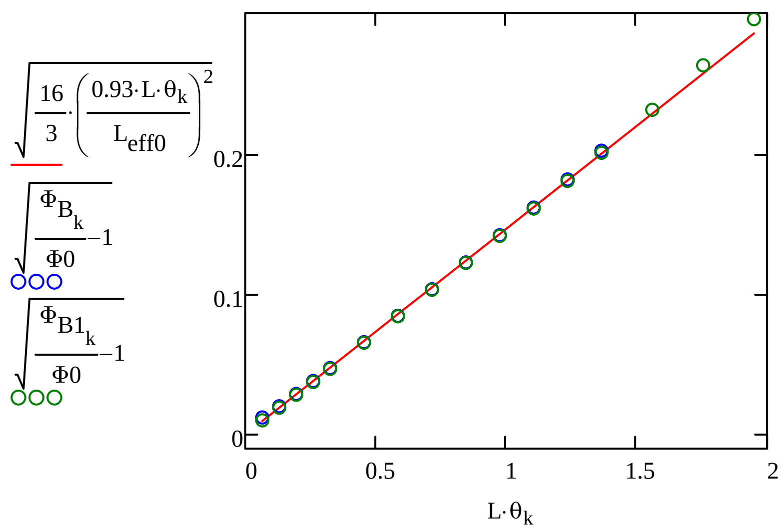




$$\mathrm{result} := \begin{pmatrix} 0.001 & 0.002 & 0.003 & 0.004 & 0.005 & 0.007 & 0.009 & 0.011 & 0.013 & 0.015 & 0.017 & 0.019 & 0.021 & 0.024 & 0.027 & 0.030 \\ 64.958 & 64.925 & 64.869 & 64.79 & 64.689 & 64.421 & 64.064 & 63.619 & 63.086 & 62.467 & 61.759 & 60.964 & 60.08 & 1 & 1 & 1 \\ 64.964 & 64.93 & 64.874 & 64.796 & 64.695 & 64.427 & 64.07 & 63.627 & 63.097 & 62.483 & 61.786 & 61.006 & 60.146 & 58.708 & 57.097 & 55.315 \\ 0.93863 & 0.93861 & 0.93859 & 0.93855 & 0.9385 & 0.93838 & 0.93821 & 0.938 & 0.93775 & 0.93745 & 0.93710 & 0.93670 & 0.93626 & 0.93551 & 0.93463 & 0.93363 \end{pmatrix} \begin{pmatrix} \theta_0 \\ F_0(N_p = 200) \\ F_0(N_p = 1000) \\ F_1(N_p = 1000) \end{pmatrix}$$

$k := 0 .. \mathrm{cols}(\mathrm{result}) - 1$ $\quad \theta_k := \mathrm{result}_{0,k}$ $\quad \Phi_{B_k} := \frac{1}{L} + \frac{1}{\mathrm{result}_{1,k}}$ $\quad \Phi_{B1_k} := \frac{1}{L} + \frac{1}{\mathrm{result}_{2,k}}$ $\quad \varphi_{b_k} := \frac{\pi}{2} - \mathrm{result}_{3,k}$ $\quad \Phi0 := \Phi_{B_1}\cdot 0.9996$





$$\Phi_0 := \frac{\kappa_B^{\,2}}{4}\cdot\int_{-L}^{L} B_F(s)^2\, ds$$

$\frac{\Phi0}{\Phi_0} = 0.951$

Numerical integration results 5% less focusing than the perturbation theory due to finite length of the lens

Conclurion: the first order perturbation theory describes well the focusing non-linearity within aperture of 13 mm

Non-linear Focusing and Resonanses in the Betatron Motion

The tune shift for linear focusing is: $\Delta\nu = \frac{1}{4\cdot\pi}\cdot\beta\cdot\Delta\Phi$

Taking into account the dependence of focusing strength or coordinate yields: $\Delta\nu = \frac{3}{16\cdot\pi}\cdot\beta\cdot\Delta\Phi$

Take into account that: $\Delta\Phi(\rho) = \frac{16}{3}\cdot\left(\frac{\rho}{L_{eff}}\right)^2\cdot\frac{1}{F}$ $\quad \frac{1}{F} = \frac{4}{L}\cdot\sin\left(\frac{\mu}{2}\right)^2$ $\quad \beta = \beta_{max} = \frac{L}{\sin(\mu)}$ $\quad \rho = \sqrt{\varepsilon\cdot\beta}$

one obtains $\Delta\nu = \frac{1}{\pi\cdot\cos\left(\frac{\mu}{2}\right)^2}\cdot\frac{\varepsilon\cdot L}{L_{eff}^{\,2}}$ ***for*** $\mu = \frac{\pi}{2}$ ***it results in*** $\Delta\nu = \frac{2}{\pi}\cdot\frac{\varepsilon\cdot L}{L_{eff}^{\,2}}$

Numerical example for Project X at injection (SSR0)

$m_p := 938\cdot10^6$ eV $\quad E_{kin} := 2.5\cdot10^6$ eV $\quad \varepsilon_n := 0.4\cdot10^{-4}$ cm $\quad \gamma := 1 + \frac{E_{kin}}{m_p}$ $\quad \beta := \sqrt{1-\frac{1}{\gamma^2}}$ $\quad \beta = 0.073$

Focusing period: $L = 65$ cm $\quad$ for $\mu = \frac{\pi}{2}$: $\quad \beta_{max} = L$

5σ emittance is: $\varepsilon_{5\sigma} := 25\cdot\frac{\varepsilon_n}{\beta\cdot\gamma}$ $\quad$ Corresponding beam size in the lens is: $\rho_{max} := \sqrt{\varepsilon_{5\sigma}\cdot\beta_{max}} = 0.943$ cm

focusing correction: $\frac{\Delta\Phi(\rho)}{\Phi} = \frac{16}{3}\cdot\left(\frac{\rho_{max}}{L_{eff0}}\right)^2 = 0.022$

and the tune shift: $\Delta\nu_{3\sigma} := \frac{2}{\pi}\cdot\frac{\varepsilon_{5\sigma}\cdot L}{L_{eff0}^{\,2}} = 2.63\times10^{-3}$ $\quad$ or $\quad \Delta\nu_{3\sigma} = 0.151\cdot deg$

Thus, the non-linearity introduced by solenoidal focusing is negligible relative to the non-linearity due to beam space charge.
If the system is detuned from the resonance (δv = 0.25 n) the focusing non-linearity does not introduce significant emittance growth. The phase advance per cell should be below 90 deg. and should not be too close to the 90 deg. to avoid the resonance.

Effect of lens non-linearity on the betatron plane rotation

For small amplitude ascillations the betatron plane rotation is:

$$\varphi_0 := \frac{\kappa_B}{2} \cdot \int_{-L}^{L} B_F(s)\, ds$$

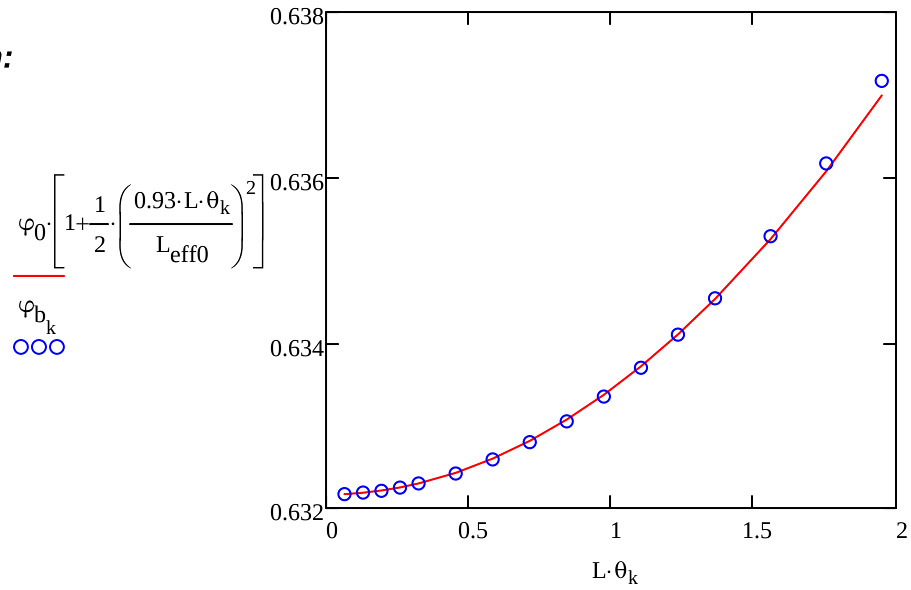
$\varphi_0 = 0.632$

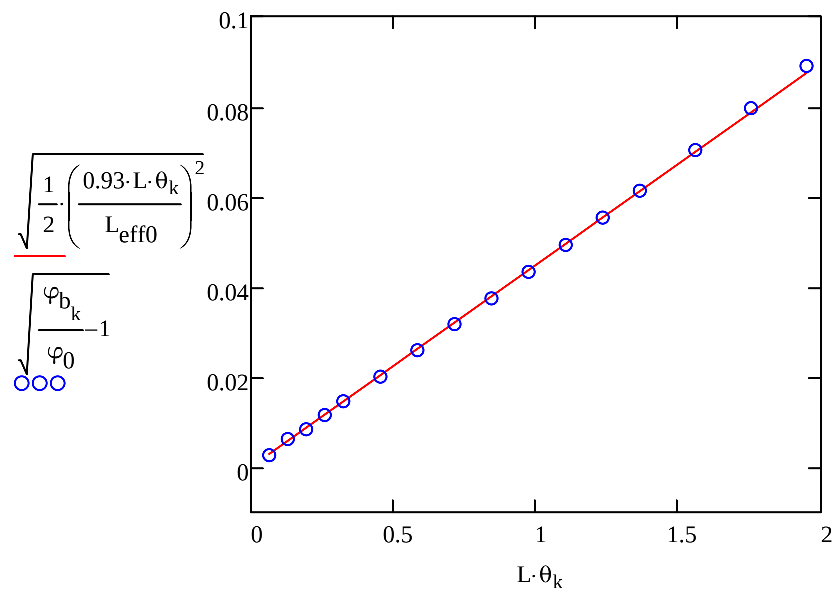
Lens nonlinearity increases the rotation of betatron plane.
For SSR0 lens it can be approximated by the following expression:

$$\frac{\Delta\varphi}{\varphi_0} = \frac{1}{2} \cdot \left(\frac{\rho}{L_{eff0}}\right)^2$$

Compare to focusing non-linearity:

$$\frac{\Delta\Phi(\rho)}{\Phi} = \frac{16}{3} \cdot \left(\frac{\rho}{L_{eff}}\right)^2$$





Steering Requirements

Focusing non-linearity creates a differenece in the horizontal and vertical focusing as well as the skew-quadrupole field. The latest results in coupling between horizontal and vertical plane and the emittance growth.

Rough estimate of the emittance growth

$$\frac{\Delta\Phi(\rho)}{\Phi} = \frac{16}{3}\cdot\left(\frac{\rho}{L_{eff}}\right)^2 \qquad \Rightarrow \qquad \delta\Phi(\rho) = \frac{16}{3}\cdot\frac{2\cdot\rho\cdot\rho_c}{L_{eff}^{\ 2}}\cdot\Phi$$

Taking into account that: $\delta\varepsilon = \frac{1}{2}\cdot\beta\cdot\delta\theta^2$ $\qquad \delta\theta = \rho\cdot\delta\Phi$ $\qquad$ ***and*** $\qquad \Phi = \frac{4}{L}\cdot\sin\left(\frac{\mu}{2}\right)^2$ $\qquad \beta = \beta_{max} = \frac{L}{\sin(\mu)}$ $\qquad \rho = \sqrt{\varepsilon\cdot\beta}$

One obtains:

$$\frac{\delta\varepsilon}{\varepsilon} = 2\left(\frac{64}{3}\right)^2\cdot\frac{\sin\left(\frac{\mu}{2}\right)^4}{\sin(\mu)^3}\cdot\frac{L\cdot\varepsilon}{L_{eff}^{\ 2}}\cdot\overline{\rho_c^{\ 2}}$$

For SSR0 with $\rho_c := 0.1$ cm we have:

$$\frac{\delta\varepsilon}{\varepsilon} = 2\left(\frac{64}{3}\right)^2\cdot\frac{\sin\left(\frac{\mu}{2}\right)^4}{\sin(\mu)^3}\cdot\frac{L\cdot\varepsilon_{5\sigma}}{L_{eff0}^{\ 2}}\cdot\overline{\rho_c^{\ 2}} = 9.402\times 10^{-3}$$ per period

Optics correction has to improve it quite significantly