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Estimations of spherical aberrations for PXIE LEBT solenoids

L.R. Prost, A. Shemyakin, S. Nagaitsev

INTRODUCTION

PXIE LEBT will transport a 5 mA, 30 keV H^- beam with several solenoidal lenses. The LEBT Functional Requirements Specification (FRS) [1] restricts the emittance growth to a maximum of 25% (taking the source emittance to be the maximum allowable value from the ion source FRS [2] i.e. $\varepsilon_n = 0.2$ mm mrad RMS normalized). One possibly major contributor to emittance growth in the LEBT is spherical aberrations from the focusing elements, namely the solenoids.

In this document, we first estimate the magnitude of these spherical aberrations from simulations of the preliminary design of a solenoid from IMP, China [3] with the SAM code [4]. Then, emittance dilution due to these aberrations is estimated.

SOLENOID GEOMETRY

Based on optics simulations of the LEBT [2], a preliminary design for the solenoids has been produced. The schematic of the solenoid from Ref. [3] is reproduced below (Figure 1).

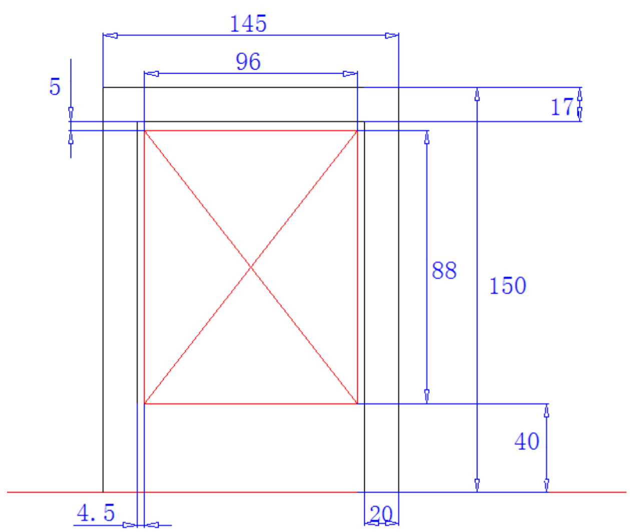

Figure 1: Schematic cross-section of the preliminary design of a LEBT solenoid. Dimensions are in mm.

The geometry above has been input into SAM and the magnetic field computed, and verified that the axial field matched the original calculations from the IMP Magnet Group design.

SIMULATIONS

Simulations were carried out for two values of the maximum magnetic field: 2.5 and 5 kG. Then, single particles starting at several offsets with respect to the center line are tracked when going through the solenoid. These particles are 30 keV protons with no initial transverse velocity.

1. Simulations with B = 2.5 kG
   a. Simulation results

Figure 2 shows the magnetic field on axis and 40 trajectories 1mm apart (in the radial direction). The particle with a 40 mm offset is 'absorbed' by the solenoid. The magnetic field is calculated from z = -400 mm to z = +500 mm. The particles are tracked over the same interval.

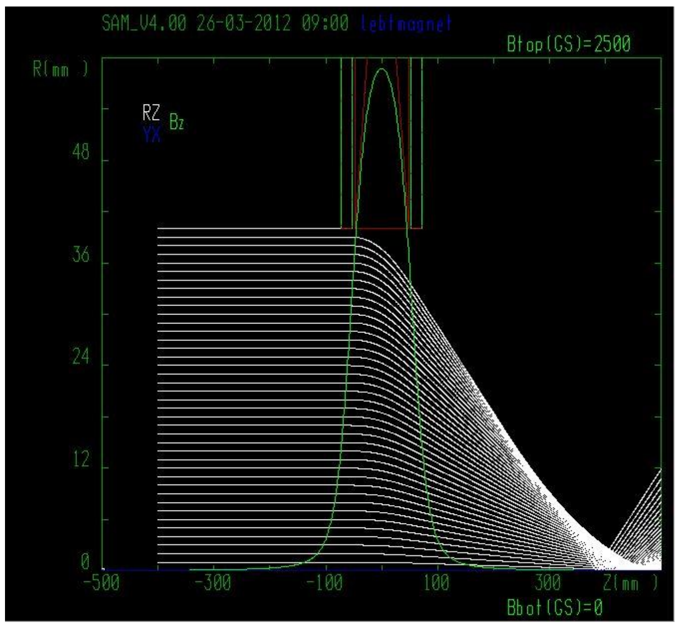

Figure 2: Field (green trace) and trajectories (white traces) for B_{max} = 2.46 kG. Note that the traces after the focal point are displayed with an upward slope only because SAM's graphical interface for axisymmetric geometries.

   b. Analysis

The spherical aberration coefficient, k, is obtained by expressing the angle of the particle exiting the solenoidal field as a function of the input radial offset, R_{in}. The corresponding equation can be written as:

$$\alpha = \frac{-R_{in}}{f_0}\left(1 + kR_{in}^2\right) \tag{1}$$

where α is the 'output' angle, and f_0 the focal length of the solenoid. Both f_0 and k are fitting parameters. Figure 3 shows the output angle versus the particle radial position before the lens calculated with SAM. The result of the fit using Eq. (1) (red trace) gives: $\frac{1}{f_0}$ **= 2.0×10^{-3} mm^{-1}** and ***k* = 1.4×10^{-4} mm^{-2}**. Note that the fit was applied only for trajectories with $0 < R_{in} < 20$ mm.

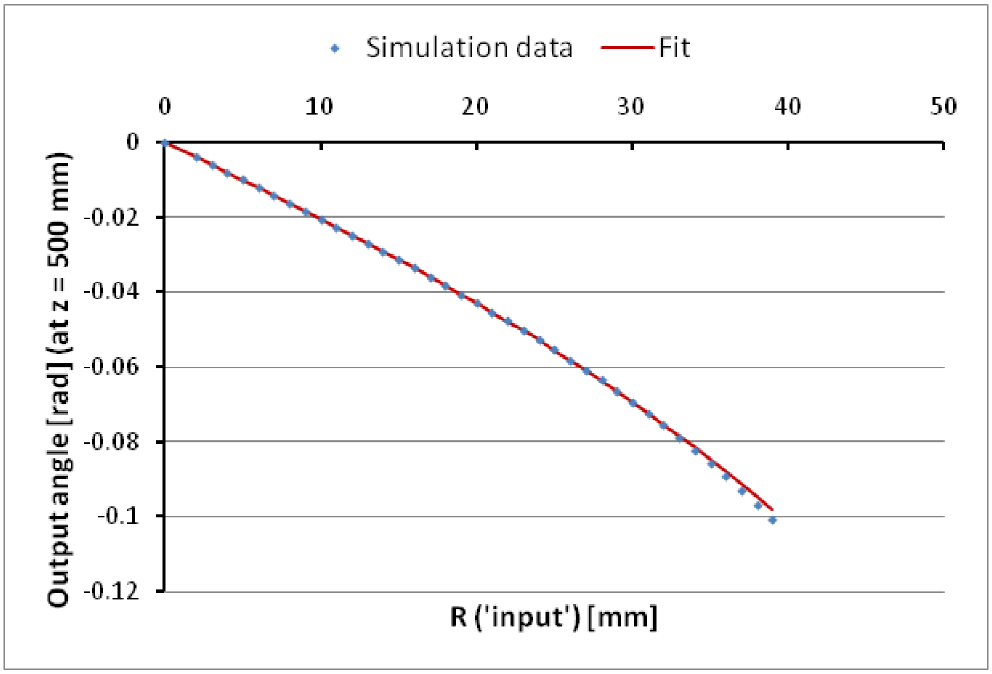

Figure 3: Output angle as a function of the particle radial position (before lens). The fit (red line) only takes into account the first 20 mm.

c. Consistency check

Since in the fit above the solenoid focal length is actually taken to be a fitting parameter, one can check that its value is consistent with a direct calculation of the focal length for the case of an optically thin lens given by:

$$\frac{1}{f_0} = \frac{q^2 \int B^2 dl}{8 \cdot E_{Kin} \cdot m_p} \tag{2}$$

where E_{kin} = 30 keV is the particle energy, m_p the proton mass, q the electric charge and $B(z)$ the magnetic field longitudinal distribution. Then, $\frac{1}{f_0}$ **= 2.1×10⁻³ mm⁻¹**, which is in agreement with the value obtained through the fitting procedure.

2. Simulations with B = 5 kG

All steps and data analyses are the same as for the B = 2.5 kG case. Figure 4 below shows the magnetic field on axis and 40 trajectories 1mm apart (in the radial direction). Figure 5 shows the output angle versus the particle radial position before the lens calculated with SAM. The result of the fit using Eq. (1) gives: $\frac{1}{f_0}$ **= 7.5×10^{-3} mm^{-1}** and ***k* = 1.4×10^{-4} mm^{-2}**. The calculation of the focal length using Eq. (2) gives $\frac{1}{f_0}$ **= 8.8×10^{-3} mm^{-1}**, which noticeably differs from the fitted value (the lens is not thin anymore).

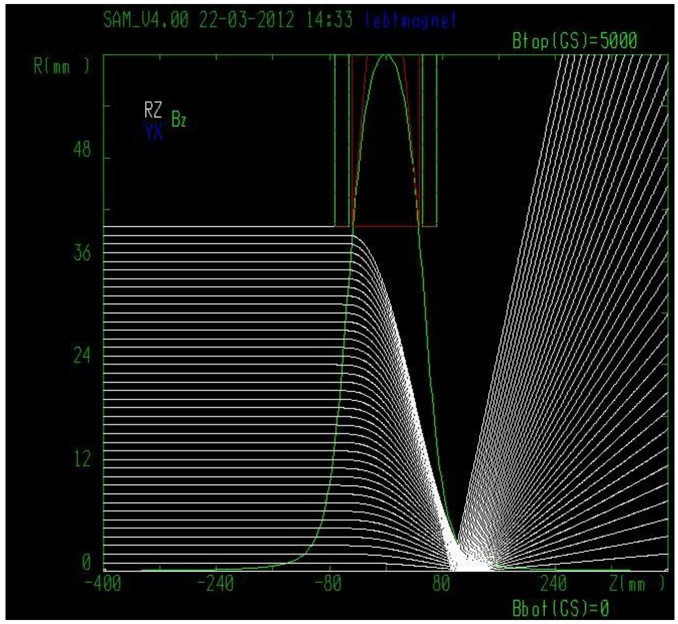

Figure 4: Field (green trace) and trajectories (white traces) for B_{max} = 5.01 kG.

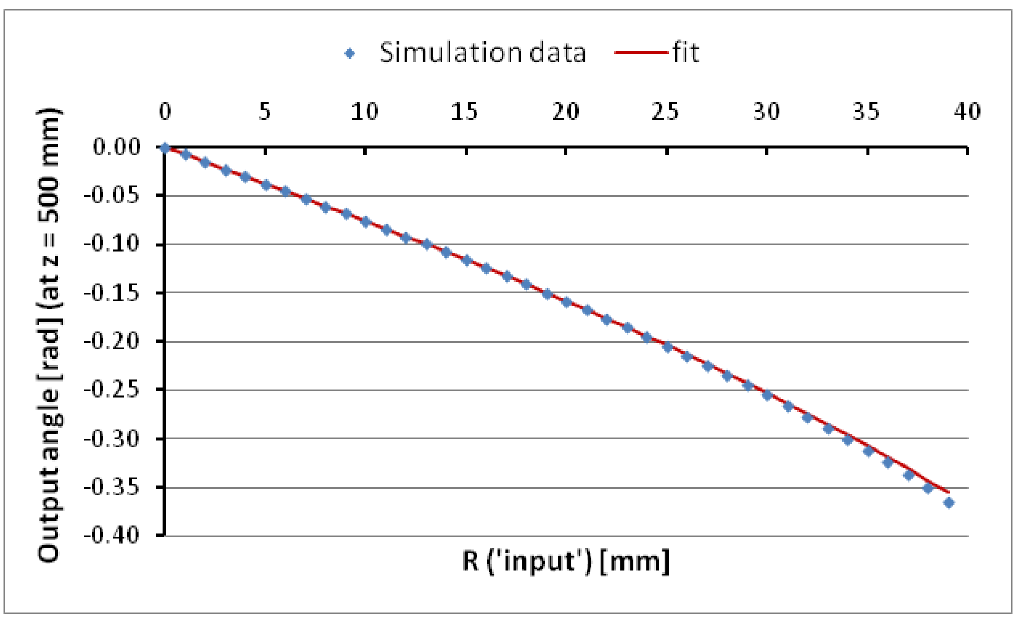

Figure 5: Output angle as a function of the particle radial position (before lens). The fit (red line) only takes into account the first 20 mm.

3. Additional comments

If all simulation data points are used for fitting, only the aberration coefficient changes; it increases by 15-20%.

Also, it should be noted that from Eq. (2), one would expect the value of $1/f_0$ for B = 5 kG to be exactly 4× the one obtained for B = 2.5 kG. However, it is not quite the case because in the computation of the magnetic field with SAM, no attempt was made to have precise values of the maximum magnetic field. Hence, the values for $\int B^2 dl$ do not differ by a factor of 4 between the two cases presented. The actual values for B_{max} are indicated in the figures captions.

COMPARISON TO AN ANALYTICAL MODEL

The magnetic field on axis can be approximated by the following equation:

$$B(z) = \frac{B_0}{2}\left(\tanh\left(\frac{z+\frac{L}{2}}{a}\right) - \tanh\left(\frac{z-\frac{L}{2}}{a}\right)\right) \quad \text{(A1)}$$

where B_0, a and L are fitting parameters. Note that these parameters are *not* the maximum magnetic field on-axis, the solenoid aperture and the solenoid length, respectively. From Eq. (A1), an analytical expression for k is obtained (see Appendix A):

$$k \equiv r_{eff}^2 = \frac{5}{2} a^2 F\left(\frac{L}{a}\right). \quad \text{(A8)}$$

The function $F(x)$ is given by Eq. (A9).

For the solenoid described in this paper, for the case B = 5 kG, one finds the following best fit: $B_0 \approx 5.5$ kG, $a \approx 35$ mm, and $L \approx 108$ mm. This, in turn, gives the following aberration coefficient: $\boldsymbol{k}$ **= 1.2×10^{-4} mm^{-2}**, which is within 20% of the value obtained with the simulation. The magnetic field on axis from Eq. (A1) is shown on Figure 6 along with the field obtained from the SAM simulation.

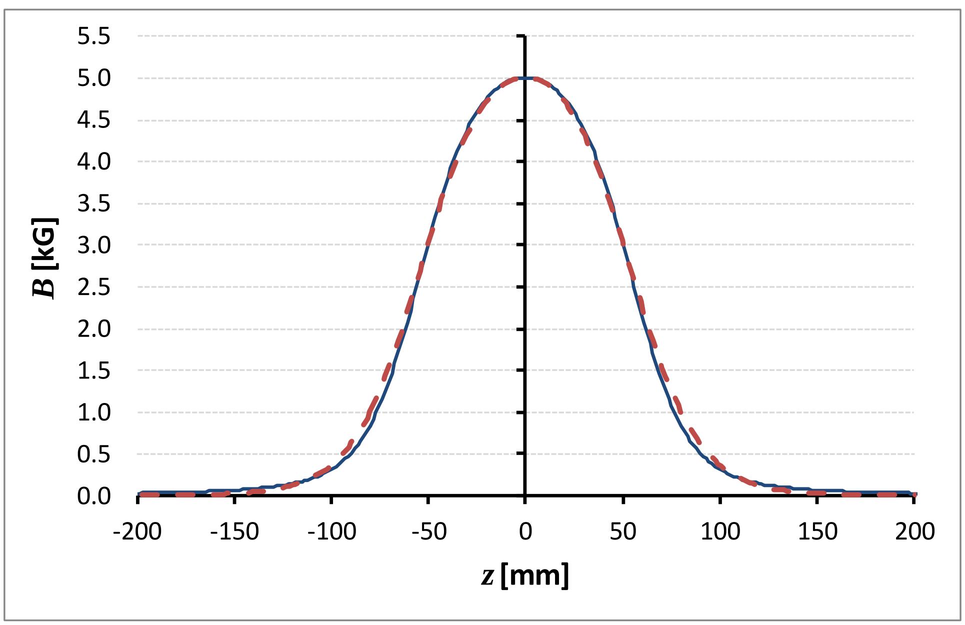

Figure 6: Magnetic field on axis for the solenoid sketched on Figure 1. Blue trace: From SAM simulation; Red trace (dotted): From Eq. (A1) with B_0 = 5.5 kG, a = 35 mm and L = 108 cm.

EMITTANCE DILUTION

For simplicity of the estimation, let's consider the case where

(i) Particle motion is uncoupled (for example, by considering it in the rotating Larmor system)
(ii) The geometry is inversed, i.e. particles start from a focal point, assuming that the aberration coefficient remains the same.
(iii) Only the first order of the aberration is considered

$$dx'(x) = -k\frac{x^3}{f_0} \equiv k_1 \cdot x^3, \tag{3}$$

where dx' and x are the aberration-related angle and offset along one of the coordinates at the output (outside of the magnetic field).

The rms emittance after the lens is

$$\varepsilon_1 = \sqrt{\overline{x^2}\cdot\overline{x'^2} - \overline{(x\cdot x')}^2}\ , \tag{4}$$

where $x' = x_0' + dx'$ is the particle's angle taking into account the angle x_0' related to its initial emittance

$$\varepsilon_0 = \sqrt{\overline{x^2}\cdot\overline{x'^2}} \equiv \sigma_{x0}\cdot\sigma_{x'0} \tag{5}$$

Assuming that the distributions without aberration are Gaussian and the spatial distribution remains Gaussian after the beam propagated through the lens, their second moments in Eq.(4) can be calculated using Eq. (3) as:

$$\begin{aligned}\overline{x'^2} &= \frac{1}{2\pi\cdot\sigma_{x0}\cdot\sigma_{x'0}}\int_{-\infty}^{\infty}\exp\left(-\frac{x_0'^2}{2\sigma_{x'0}^{\ 2}}\right)dx'\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2\sigma_{x0}^{\ 2}}\right)\left(x_0' + k_1\cdot x^3\right)^2 dx = \sigma_{x'0}^{\ 2} + 15k_1^{\ 2}\cdot\sigma_{x0}^{\ 6};\\ \overline{(x\cdot x')} &= \frac{1}{2\pi\cdot\sigma_{x0}\cdot\sigma_{x'0}}\int_{-\infty}^{\infty}\exp\left(-\frac{x_0'^2}{2\sigma_{x'0}^{\ 2}}\right)dx'\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2\sigma_{x0}^{\ 2}}\right)\left(x\cdot x_0' + k_1\cdot x^4\right)dx = 3k_1\cdot\sigma_{x0}^{\ 4};\end{aligned} \tag{6}$$

Then, combining equations (4)-(6), the final emittance can be written as:

$$\varepsilon_1 = \sqrt{\sigma_{x0}^{\ 2}\cdot\sigma_{x'0}^{\ 2} + 6k_1^{\ 2}\sigma_{x0}^{\ 8}} = \varepsilon_0\sqrt{1 + 6\frac{\delta a^2}{\sigma_{x'0}^{\ 2}}}\ , \tag{7}$$

where $\delta a \equiv k_1\sigma_{x0}^{\ 3}$ is the aberration angle at the rms offset. For the emittance $\varepsilon_0 = 25\,\mu m$ (calculated from $\varepsilon_n = 0.2\,\mu m$), the values for k and $1/f_0$ obtained from the lens simulation with B = 5 kG, and the rms beam size of 7 mm, the rms emittance dilution estimated with Eq. (7) is ${}^{\varepsilon_1}/_{\varepsilon_0} - 1 \approx$ 3%.

DISCUSSION

The estimated value of the spherical aberration calculated above does not contribute significantly to the emittance budget for the parameters of the simulated lens. While for real lenses higher order aberrations can significantly exceed the spherical component (e.g.: [5]), the estimation likely gives the correct order of magnitude of possible aberrations introduced during the fabrication.

Bibliography

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APPENDIX A

The magnetic field on axis is approximated by the following 3-parameter expression:

$$B(z) = \frac{B_0}{2}\left(\tanh\left(\frac{z+\frac{L}{2}}{a}\right) - \tanh\left(\frac{z-\frac{L}{2}}{a}\right)\right), \tag{A1}$$

where a is the aperture parameter and L is the length parameter. This approximation works well for many solenoids if $L \geq 2\times$**Aperture**. To determine the appropriate parameters for a realistic solenoid one needs to know the maximum magnetic field, B_{max}, and the two integrals: $\int B dz$ and $\int B^2 dz$. The solenoid parameters (B_0, L and a) are then the solutions of the following system of algebraic equations:

$$B_{max} = B_0 \tanh\left(\frac{L}{2a}\right), \tag{A2}$$

$$\int B(z)^2 dz = B_0{}^2 a\left(\frac{L}{a}\coth\left(\frac{L}{a}\right) - 1\right), \tag{A3}$$

$$\int B(z) dz = B_0 L\,. \tag{A4}$$

The spherical aberration can be expressed by the following simplified formula:

$$f^{-1}(r) = f_{sol}^{-1}\left(1 + \frac{r^2}{r_{eff}^2}\right), \tag{A5}$$

where

$$f_{sol}^{-1} = \frac{1}{4}\left(\frac{e}{pc}\right)^2 \int_{-\infty}^{\infty} B(z)^2 dz \tag{A6}$$

is the paraxial focal length, and

$$r_{eff}^2 = \frac{2\int_{-\infty}^{\infty} B(z)^2 dz}{\int_{-\infty}^{\infty} B'(z)^2 dz} \tag{A7}$$

is the aberration coefficient.

An approximate expression (with the <5% accuracy and the correct asymptotes) for the aberration coefficient is:

$$r_{eff}^2 = \frac{5}{2} a^2 F\left(\frac{L}{a}\right), \tag{A8}$$

where

$$F(x) \approx \frac{6\left(x\coth(x)-1\right)}{5\tanh\left(\sqrt{\frac{2}{5}}x\right)^{2}}. \tag{A9}$$

For $L \gg a$, the aberration coefficient is

$$r_{eff}^{2} \approx 3a\left(L-a\right). \tag{A10}$$

Equation (A9) is plotted below.

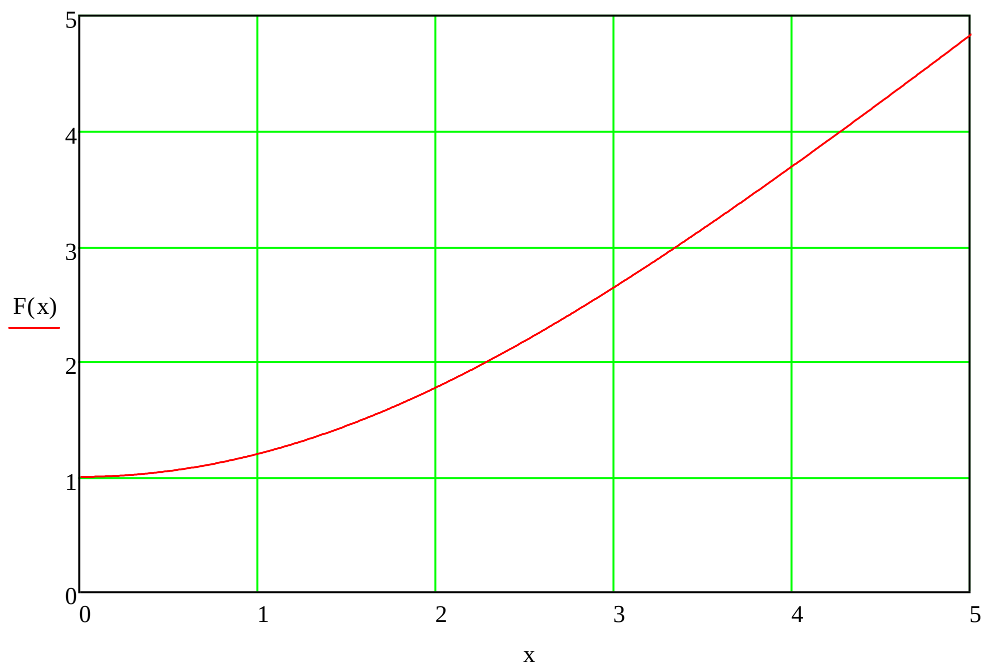

Figure A.1: Function $F(x)$.